Design and Development of Strip-Line Filters

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Summary—Strip transmission lines offer an alternate medium in which microwave filters can be realized. Since bandpass filters designed in waveguide or coaxial lines would necessarily be large at ultra-high frequencies, strip lines provide a practical means of realizing filters which are simply fabricated and which represent an appreciable saving in size and weight.

Design techniques which were formerly employed in the realization of waveguide and coaxial filters have been applied in the synthesis of strip-line filters having "maximally-flat" and Tchebycheff response characteristics. In this paper, these techniques as well as those for realizing the required circuit parameters in strip line will be described. Using engraving techniques instead of the more commonly employed photo-etching techniques, strip-line filters can be fabricated with such precision that tuning screws are generally not required to align the filters to a desired center frequency. Bandpass filters having a 10 per cent bandwidth at S-band have been developed with a mid-band insertion loss of less than 1 db and a rejection of greater than 50 db at frequencies 9 per cent from the center frequency. The effect of temperature variations on the filter performance has been evaluated and will be discussed. Techniques will also be presented for eliminating spurious responses in filters required to operate over a frequency range of several octaves.

The application of strip-line techniques in the development of multiplexers and other microwave components, such as detector mounts, attenuators, and loads, will be discussed.

INTRODUCTION

SINCE World War II, technological advances in the utilization of ultra-high frequencies in the fields of communications, navigation, air surveillance and television have promoted an increasing interest in the development of rf filters designed to provide frequency discrimination and signal selection in both airborne and ground based equipment. Since such components designed in waveguide and coaxial lines would be large in the uhf spectrum, a relatively new distributed line structure, namely, strip-line, is particularly well suited for the realization of filters which are simply fabricated, are readily reproduced, and, in most cases, represent an appreciable saving in size and weight.

The purpose of this paper is, in general, to acquaint the engineer with some of the advantages afforded in the realization of rf components in a strip-line structure at ultra-high frequencies, and, more specifically, to present a general design procedure which can be employed in strip-line for the development of bandpass filters having a prescribed bandwidth, skirt selectivity, and pass-band ripple tolerance. In a previous paper,[1] design techniques for waveguide and coaxial filters,[2–4] were applied in the synthesis of strip-line filters having a "maximally-flat" selectivity. In this paper, a more general procedure is presented from which strip-line filters can be designed having either "maximally-flat" or Tchebycheff characteristics. However, as a result of the mathematical simplicity of the design equations in the former paper, the formulas presented here are recommended primarily for the use in the design of bandpass filters having a prescribed ripple tolerance or Tchebycheff response.

Experimental strip-line filters having 10 per cent bandwidths at S-band have been developed with less than 1 db mid-band insertion loss and with greater than 50 db rejection at frequencies 9 per cent from the center frequency. Using engraving techniques instead of the more commonly employed photo-etching techniques[5,6] strip-line filters can be fabricated with such precision that tuning screws are generally not required to align the filters to a desired center frequency. The design techniques discussed in this paper are general and, therefore, not restricted to the realization of the above filter characteristics. The application of strip-line techniques in the development of other microwave components such as detector mounts, attenuators and loads will also be discussed.

FILTER DESIGN PROCEDURE

Bandpass filters to be realized in a strip-line structure can be synthesized in a manner similar to that employed in the design of lumped parameter filters at much lower frequencies.[7,8] This close correlation between lumped and distributed parameter circuits is not always recognized. For that reason, it is hoped that the following design procedure will be both enlightening and helpful to those engineers who are interested in the development of bandpass filters in a strip-line structure.

The first step in the design of filters is the selection of an appropriate power loss ratio, P_0/P_L, or transmission coefficient, $|t(j\omega)|^2$, corresponding to a network having the desired rejection-band characteristics. One type of commonly used transmission coefficient, which results in what is called a "maximally-flat" selectivity, is that derived from the Butterworth function:

$$|t(j\omega)|^2 = \frac{1}{1+\omega^{2n}}, \tag{1}$$

† Melpar, Inc., Falls Church, Va.

[1] E. H. Bradley and D. R. White, "Band-pass filters using strip-line techniques," *Electronics*, vol. 25, pp. 152–155; May, 1955.

[2] W. W. Mumford, "Maximally-flat filters in waveguide," *Bell. Sys. Tech. Jour.*, vol. 27, pp. 684–713; October, 1948.

[3] G. C. Southworth, "Principles and application of waveguide transmission," D. Van Nostrand Co., Inc., New York, N. Y., p. 297; 1950.

[4] G. L. Ragan, "Microwave transmission circuits," Rad. Lab. Ser., McGraw-Hill Book Co., New York, N. Y. pp. 645–673; 1948.

[5] R. M. Barrett, "Etched sheets serve as microwave components," *Electronics*, vol. 25, pp. 114–118; June, 1952.

[6] K. S. Packard, "Machine methods make strip transmission line," *Electronics*, vol. 27, pp. 148–150; September, 1954.

[7] E. A. Guillemin, "Modern methods of network synthesis," *Advances in Electronics*, vol. III, Academic Press, Inc., New York, N. Y. pp. 261–303; 1951.

[8] J. R. Whinnery, "Design of microwave filters," *Proc. Sym. Mod. Network Synth.*, New York, pp. 296–311; April, 1952.

where n designates the number of frequency sensitive elements in the low-pass prototype shown in Fig. 1 and ω is the normalized frequency variable in radians. Still another type of transmission coefficient, which is based on the Tchebycheff polynomial $T_n(j\omega)$ and causes an oscillatory behavior in the pass-band, is expressed in (2):

$$|t(j\omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^{\,2}(\omega)}, \tag{2}$$

where ϵ specifies the ripple tolerance in the pass-band. The Butterworth function, whose poles are uniformly distributed on a semicircle centered at the origin of the complex frequency plane, is classified as a special case of the Tchebycheff function, in which the poles lie on the periphery of a semi-ellipse centered at the origin. For given band-rejection characteristics or skirt selectivity, fewer elements are required when the synthesis is based on a Tchebycheff distribution.

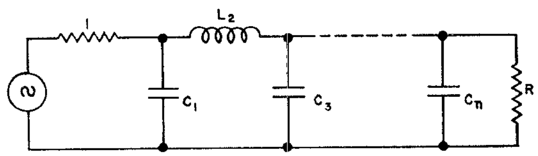

Fig. 1—Low-pass filter prototype.

In the design of filters requiring extremely sharp skirt selectivity, the Jacobian elliptic function is often used to approximate the desired response function. However, for most practical applications, the use of either the Butterworth or Tchebycheff approximation will suffice.

Following the selection of an appropriate transmission coefficient, which for the purpose of this paper is assumed to be either the Butterworth or Tchebycheff response function, the corresponding low-pass prototype can be synthesized from (1) or (2) using a procedure outlined in Appendix A. From this low-pass prototype, a bandpass filter [*c.f.* Fig. 2(a)] can be readily obtained with a low-pass-to-bandpass transformation which is equivalent to replacing each series inductance of the prototype with a series LC circuit resonant at the desired center frequency and each shunt capacitance with a parallel resonant tank. As will be seen later, this configuration is not as easily realized in strip-line as is the circuit shown in Fig. 2(b).

It should be noted that the transmission coefficient of the former network has an equal zero distribution at zero and infinite frequencies in the complex frequency plane, whereas the filter shown in Fig. 2(b) has an unequal distribution with $2n-1$ zeros at zero frequency and 1 at infinite frequencies (n is here defined as the number of resonant elements). For this reason, there exists no one-to-one relationship between the response characteristics of these two circuits. However, if the driving-point impedances of the two networks are equated and if the circuit components of one network is expressed in terms of the other, the pole distribution of each network will be identical. Although the zero distribution of the filters differ, the response characteristics are essentially identical in the neighborhood of the passband for small filter bandwidths. For large bandwidths, the frequency response characteristic of the filter having an unequal zero distribution will be asymmetrical.

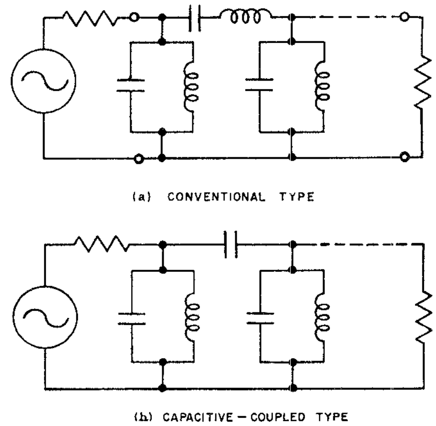

Fig. 2—Lumped element bandpass filters.

By equating the driving-point impedances for the band-pass equivalent of the low-pass prototype [*c.f.* Fig. 2(a)] and the capacitive-coupled filter shown in Fig. 2(b), the circuit parameters of the latter network can be expressed in terms of those of the prototype.[9] In order to design an equivalent filter in strip-line, an analogy can be made between the capacitive-coupled filter and the direct-coupled cavity-type filter of Fig. 3.

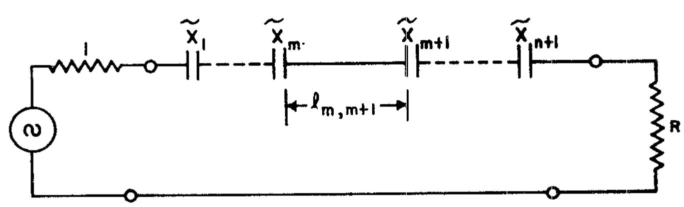

Fig. 3—Direct-coupled cavity-type filter.

Considering the first resonant circuit as an asymmetrically loaded cavity, the normalized coupling reactance at the input, $\tilde{X}_1$, can be expressed in terms of the prototype as follows:

$$|\tilde{X}_1| \simeq \sqrt{\frac{2}{\pi} C_1 \frac{f_0}{\Delta f}}, \tag{3}$$

where C_1 is derivable from the prototype (*c.f.* Fig. 1)

f_0 = center frequency of the bandpass filter
Δf = filter bandwidth.

In the design of filters having bandwidths in excess of 10 per cent, the accuracy of (3) is questionable.

[9] Ragan, *op. cit.*, pp. 661–666.

The normalized reactance of the end discontinuity is

$$|\tilde{X}_{n+1}| = |\tilde{X}_1| \sqrt{\frac{Ln}{RC_1}} \quad \text{for } n \text{ even} \tag{4}$$

and

$$|\tilde{X}_{n+1}| = |\tilde{X}_1| \sqrt{\frac{RC_n}{C_1}} \quad \text{for } n \text{ odd,} \tag{5}$$

and the normalized reactances of all other discontinuities are

$$|\tilde{X}_m| = |\tilde{X}_1|^2 \frac{\sqrt{L_{m-1}C_m}}{C_1} \quad \text{for } m \text{ odd but } > 1, \tag{6}$$

and

$$|\tilde{X}_m| = |\tilde{X}_1|^2 \frac{\sqrt{L_m C_{m-1}}}{C_1} \quad \text{for } m \text{ even,} \tag{7}$$

where

m = serial number of the discontinuity = 1, 2, 3, $\cdots \leqq \eta$

R, L_m, C_m = prototype parameters designated in Fig. 1.

The separation of the reactance elements is

$$l_{m,m+1} = \frac{\lambda\alpha}{2\pi\sqrt{\epsilon}} \cdot \left[k\pi - \frac{1}{2}\left(\tan^{-1}\frac{2}{|\tilde{X}_m|} + \tan^{-1}\frac{2}{|\tilde{X}_{m+1}|}\right)\right], \tag{8}$$

where

$\lambda\alpha$ = electrical wavelength in air

k = a positive integer

ϵ = dielectric constant of the transmission medium.

Physical Realization

Following the design of the bandpass filter, there remains the difficult problem of physically realizing the required circuit components in a strip-line structure. In order to achieve this objective, it is necessary to 1) choose the type of line structure, 2) select the most appropriate dielectric material for the particular application, 3) compile a reference library of normalized capacitive reactances vs gap spacing, and 4) fabricate the multi-stage filter using photo-etching and/or engraving techniques.

There are several different type strip transmission line structures,[10,11] the better known of which are illustrated in Fig. 4. In the development of rf components where line discontinuities are required, Melpar has employed the "sandwich" structure in order to reduce those problems introduced by line radiation and to facilitate the packaging of components. The choice of an air-dielectric or solid-dielectric structure is predicated largely upon the system requirements regarding space and weight as well as upon the allowable midband insertion loss.

In general, strip-line filters designed to operate at high frequencies (*i.e.*, $f_0 > 3{,}000$ mc) should be designed in an air-dielectric structure such as shown in Fig. 4(d). At lower frequencies where the consideration of space is often important, similar components can be designed in a solid-dielectric structure whose over-all size is reduced by the square root of the dielectric constant; in so doing, however, the weight of the units is appreciably increased. In the development of narrow bandwidth filters (*e.g.*, bandwidth less than 1 per cent), the presence of a solid-dielectric material between the strip and the associated ground planes generally increases the midband insertion loss of the assembly to an extent dependent upon the frequency of operation. When a solid-dielectric structure is desired, the line shown in Fig. 4(c) is particularly advantageous. Although the single-line structure [*c.f.*, Fig. 4(b)] can be fabricated more readily, the use of a double-strip structure greatly simplifies the fabrication of other strip-line components, such as attenuators, loads, detector mounts, and tunable filters. Although certain general recommendations can be made, the choice of an air or solid-dielectric structure must be consistent with the specific electrical and physical specification of the component. In order to facilitate the development and interconnection of strip-line components, the line structure shown in Figs. 4(c) and 4(d) have been standardized at Melpar.

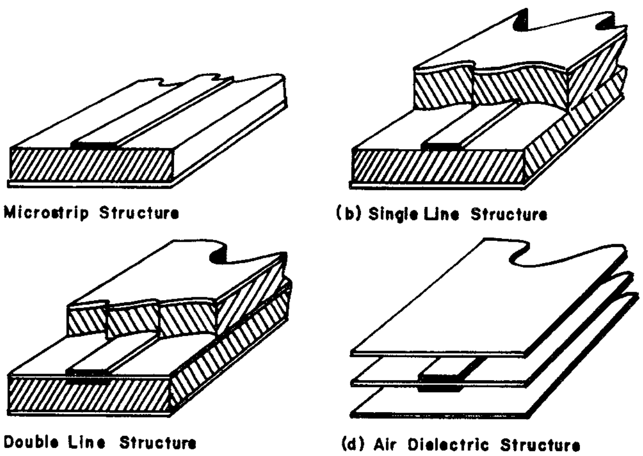

Fig. 4—Strip transmission line structures.

The choice of a dielectrical material for use in the strip-line structure is particularly important for several reasons:

1) Variations in the wavelength will drastically effect the electrical performance of the filter.
2) A lossy dielectric material will appreciably increase the midband insertion loss and limit the maximum loaded Q which can be realized.

For these reasons, only those materials having a low loss-tangent, a satisfactory degree of homogeneity, suitability for copper cladding and good environmental

[10] E. G. Fubini, W. E. Fromm, and H. S. Keen, "New techniques for high-Q strip components," Convention Record of the IRE, Part 8, p. 91; 1954.

[11] D. D. Greig and H. F. Englemann, "Microstrip—A new transmission technique for the kilomegacycle range," Proc. IRE, vol. 40, pp. 1644–1650; December, 1952.

characteristics can be considered. Following an extensive investigation of various copper-clad dielectric materials (*viz.*, Mycalex, pure teflon, epoxy, ceramic, and teflon fiberglass), teflon fiberglass has been selected for use at Melpar.

The final preparatory step in the realization of the capacitive-coupled strip-line filters is the compilation of normalized reactance data. A series capacitance is easily realized in a strip transmission line by making a gap in the center conductor [*c.f.*, Fig. 5(a)]. A reference library of normalized capacitive reactances as a function of gap spacing can be compiled experimentally in either of two ways:

1) Test one-stage strip-line filters having different gap spacings at a specified frequency.
2) Make impedance measurements on a single gap using slotted-line techniques.

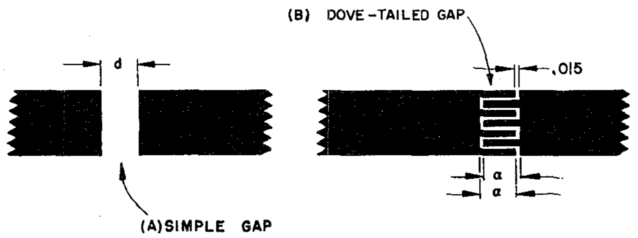

Fig. 5—Series capacitance realization in strip line.

The former approach has been employed at Melpar since the capacitive reactance of a gap spacing can be readily computed from (9) following the measurement of the loaded Q of a one-stage filter:

$$|\tilde{X}_g| = \sqrt{\frac{4}{\pi}\frac{f_0}{\Delta_f}}. \quad (9)$$

In order to realize the higher capacitances required at the lower frequencies without making the gap spacings unrealistically small (less than 0.010 inches), a dove-tailed gap shown in Fig. 5(b) is employed. Typical reference libraries of standard and dove-tailed gaps compiled at 2,000 mc in an air-filled line are shown in Fig. 6.

In an effort to maintain closer tolerances in the fabrication of strip-line filters, an engraving technique has been developed to replace the less accurate photo-etching methods still used on less precise work. An experimental evaluation of the reproducibility of strip-line filters using both fabrication techniques has verified that improved tolerances can be realized with engraving. Using photo-etching techniques, the widths of the smallest capacitive gaps employed (viz., 0.010 inch) were found to vary from −35 per cent to +5 per cent, while the variation in similar engraved gaps can be maintained within micrometer tolerances. Although the bandwidth on the photoetched cavities ($Q_L=40$) centered at 3,000 mc varied as much as 19 per cent, the bandwidth of the engraved cavities varied less than 6 per cent.

Experimental Results

Using the formulas and reference data discussed above, bandpass filters having either Tchebycheff or "maximally-flat" response characteristics can be designed and fabricated. The choice of a particular filter selectivity is dependent upon the desired application.

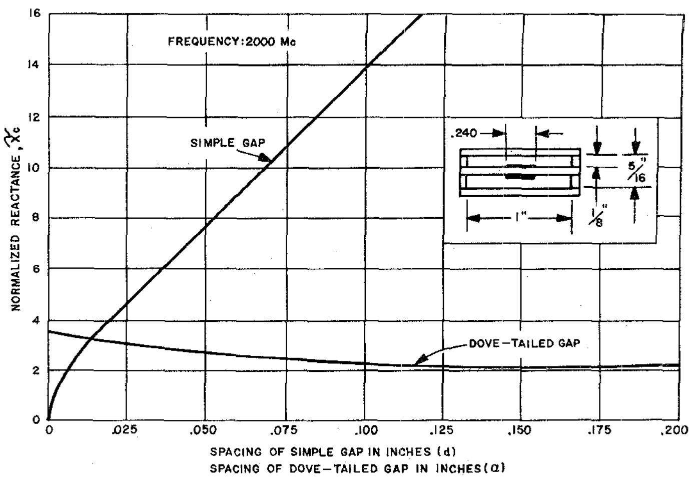

Fig. 6—Design data for strip-line filters.

Five-stage bandpass filters having a Tchebycheff response function and a 3-db ripple tolerance have been developed in an air-dielectric structure at 2,150 mc with a 10 per cent bandwidth. A typical unit shown in Fig. 7 has a minimum insertion loss of less than 1 db,

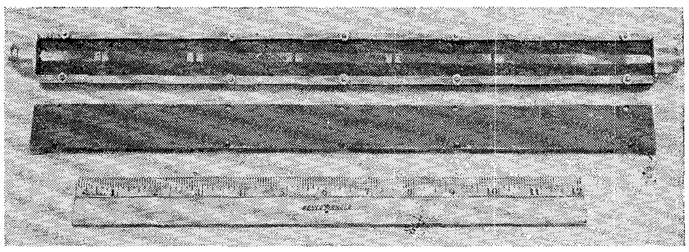

Fig. 7—Five-stage filter having Tchebycheff response.

and a rejection of greater than 50 db at frequencies 9 per cent from the center frequency; the theoretical and experimental response functions are plotted in Fig. 8.

For those applications where ripples in the filter pass-band must be kept to a minimum, strip-line filters can be designed to have a "maximally-flat" selectivity using the same design equations. Six-stage "maximally-flat" filters having a 10 per cent bandwidth and 40 db rejection at frequencies 12 per cent from the center frequency have been developed at 1,435 mc. Typical theoretical and experimental response functions for such a solid-dielectric filter are shown in Fig. 9.

It is of interest to note that the performance of strip-line filters under extreme environmental operating conditions is particularly good. In temperature cycling a six-stage filter with an air-dielectric structure from

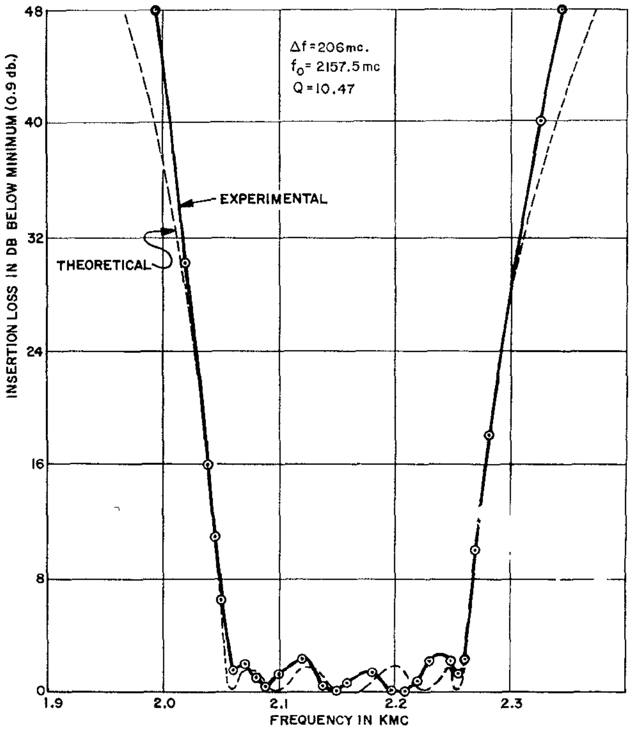

Fig. 8—Tchebycheff response characteristics of five-stage strip-line filter.

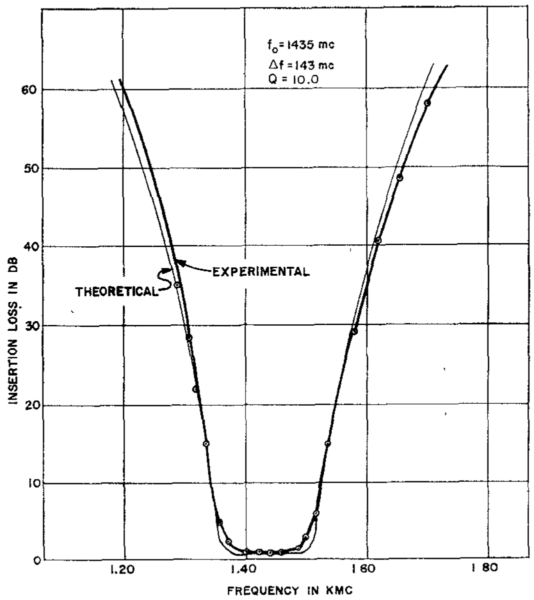

Fig. 9—Maximally-flat response characteristics of six-stage strip-line filter.

+25°C to +125°C, the center frequency decreased only 0.03 per cent at 125°C corresponding to a 0.3 db change in the insertion loss at the 8 db points on the filter skirts.

Bandpass filters having both Tchebycheff and "maximally-flat" response functions are currently being developed for use in the frequency range from 400 to 6,000 mc. In order to shorten those filters designed at the lower frequencies, the strip-line can be fabricated in a "snake-like" configuration with reactance gaps or the equivalent lumped capacitors located in the linear portion of the line. The effect of reflections from the impedance discontinuities at the bends has been found to be negligible.

In many applications, filters must have a large insertion loss over a wide frequency spectrum outside of their fundamental pass-bands. However, distributed parameter networks, similar to those discussed here, are subject to spurious responses at approximately integer multiples of the fundamental. Spurious responses of this type can be removed with a low-pass filter having a cutoff frequency somewhat less than that of the first spurious response; this type of filter can be readily developed in a strip-line structure.[12]

Associated Strip-Line Components

Strip-line is not only well-suited for the realization of bandpass filters but also for the fabrication of numerous other associated rf components,[13] such as tunable detector mounts, pad attenuators, rf loads, and multiplexers.

For example, tunable detector mounts having bandwidths greater than 10 per cent have been designed to tune over a frequency range from 400 mc to greater than 6,000 mc. A typical unit, shown in Fig. 10, has a sensitivity of greater than −48 dbm over the designed frequency range when operated in conjunction with a conventional video amplifier. These mounts have been developed in both solid and air-dielectric structures.

Fixed attenuators having insertion losses of 3 to 10 db have been developed with vswr's of less than 1.3 over a 2 to 1 frequency range centered at S-band using 50-ohm resistive cards placed symmetrically between the strip and the adjacent ground planes. Using similar techniques, strip-line loads have been developed for use over a comparable frequency range.

Since most laboratory test equipment is designed for use with either waveguide or coax structures, the development of both coax-to-strip-line and waveguide-to-strip-line adaptors was required. Broadband transitions for both waveguide and coax have been developed and are show in Fig. 11; the vswr of each of these units is less than 1.3 over its operation frequency range. Flange connectors are utilized to facilitate the interconnection of strip-line components without requiring the use of intermediate adaptors. In order to facilitate the evaluation of these components, a strip-line slotted section operating in a standard Hewlett Packard carriage has been designed to operate over a frequency range from 2,000 to 6,000 mc; this unit is shown in Fig. 12.

[12] H. C. Hyams, "Design of a flat strip or printed circuit microwave low-pass filter," *Melpar Tech. Rep.*

[13] N. R. Wild, "Photoetched microwave transmission lines," *Symposium on Microwave Strip Circuits*, Tufts College; October 1954.

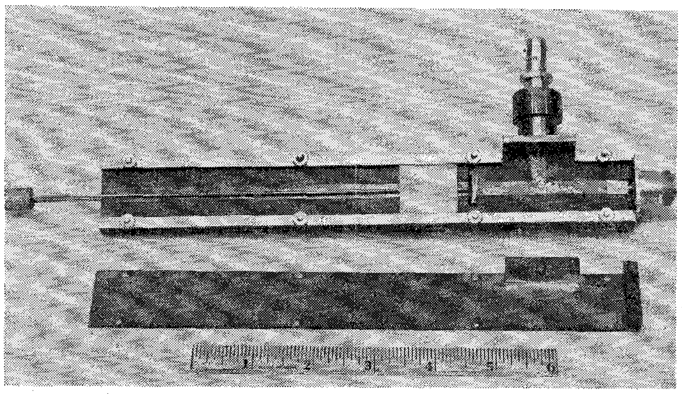

Fig. 10—Tunable strip-line detector mounts.

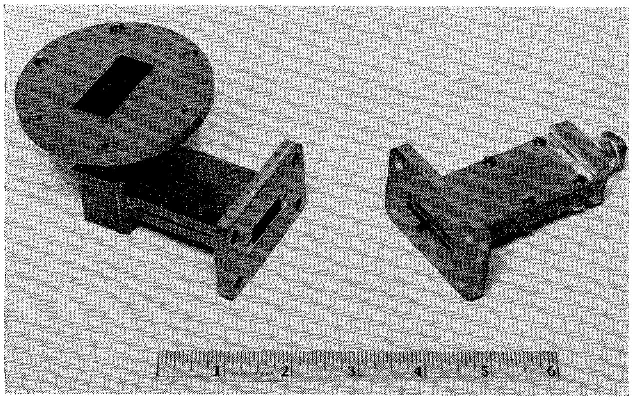

Fig. 11—Strip-line transitions to waveguide and coaxial line.

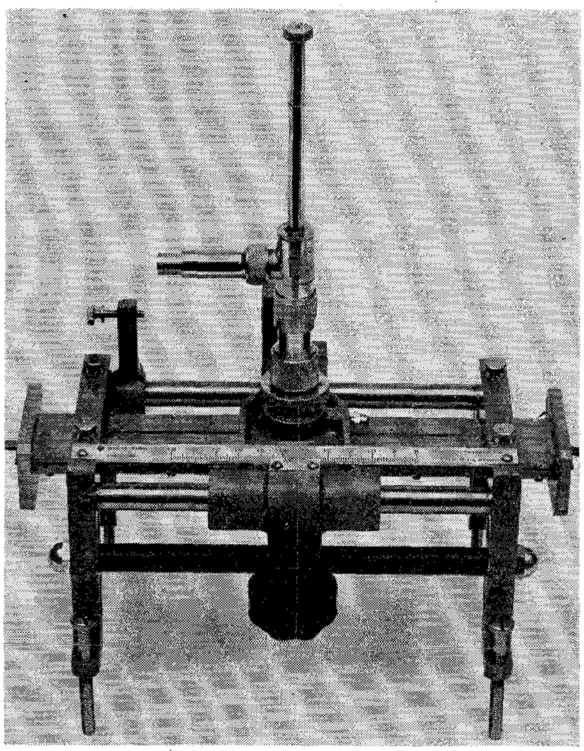

Fig. 12—Strip-line slotted section.

Conclusion

The design formulas and the realization techniques presented here provide the engineer with a formalized design procedure for the development of bandpass filters having a prescribed bandwidth, skirt selectivity and passband ripple tolerance in a strip-line structure. Rf components developed in strip line have certain advantages over their waveguide and coaxial line counterparts in the uhf spectrum. For that reason, it is anticipated that strip-line techniques will be found increasingly applicable in this frequency range for the future development of rf components, such as bandpass filters.

Appendix A

Synthesis of Two-Terminal-Pair Networks Terminated at Both Ends[14]

The power-loss ratio, P_0/P_L, or the transmission coefficient, $|t(j\omega)|^2$, is used to specify the insertion loss characteristics of a two-terminal-pair network terminated at both ends. Mathematically, this rational function or ratio of two polynomials can take any one of many different forms depending upon the desired frequency response. The Butterworth function (sometimes called the "maximally-flat" function), the Tchebycheff functions of the first and second kind, and the Jacobian elliptic function are commonly used in the design of amplifiers, receivers and filters.

A low-pass prototype having resistive terminations at both terminal pairs can be synthesized[15] such that its response function approximates the Tchebycheff polynomial of the first kind having a ripple tolerance ϵ, or

$$|t(j\omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\omega)}. \tag{10}$$

where

$t(j\omega)$ = ratio of the voltage of the transmitted wave to the voltage of the incident wave.

ϵ = ripple tolerance[16]

$T_n(\omega)$ = Tchebycheff polynomial of the first kind.[17]

Furthermore,

$$|t(j\omega)|^2 = 1 - |\rho(j\omega)|^2 \tag{11}$$

[14] The following appendix is not intended to give a rigorous development of a synthesis technique, but rather to formulate a workable procedure which can be readily applied by the engineer.

[15] E. A. Guillemin, unpublished class-room notes.

[16] The ripple tolerance in db is $-10 \log (1+\epsilon^2)$.

[17] The Tchebycheff polynomial, $T_n(\omega)$, can be expanded as follows:

$$T_1(\omega) = \omega$$
$$T_2(\omega) = 2\omega^2 - 1$$
$$T_{n+1}(\omega) = 2\omega T_n(\omega) - T_{n-1}(\omega).$$

or

$$|\rho(j\omega)|^2 = \rho(j\omega)\rho(-j\omega) = \frac{\epsilon^2 T_n^2(\omega)}{1+\epsilon^2 T_n^2(\omega)}, \qquad (12)$$

where

$$\rho(j\omega) = \text{reflection coefficient.}$$

To insure physical realizability, the poles of the function expressed in (12) must be in the left half of the complex frequency plane (*i.e.*, $\sigma_n \leqq 0$ where the complex variable, s, is defined as $s_n = \sigma_n + j\omega_n$). The zeros of the function are not so restricted; however, in order to maximize the gain-bandwidth product for a given associated shunt capacitance, the zeros must lie in the left half-plane.

For the purpose of this discussion, a minimum phase function (*i.e.*, zeros in the left half-plane) will be formulated. Using the following equations, the poles and zeros can be immediately identified:

$$s_{pk} = -\sin\frac{k\pi}{2n}\sinh\phi_2 + j\cos\frac{k\pi}{2n}\cosh\phi_2, \qquad (13)$$

where

S_{pk} = location of the pole in the complex plane.
$k = 1, 3, 5, \cdots 2n-1$.
n = order of the polynomial (*i.e.*, number of frequency sensitive elements in the low-pass structure).

$$\sinh\phi_2 = \frac{(\alpha+\beta)^{1/n} - (\alpha-\beta)^{1/n}}{2} \qquad (14)$$

$$\cosh\phi_2 = \frac{(\alpha+\beta)^{1/n} + (\alpha-\beta)^{1/n}}{2} \qquad (15)$$

$$\alpha = \sqrt{1+\frac{1}{\epsilon^2}} \qquad (16)$$

$$\beta = \frac{1}{\epsilon} \qquad (17)$$

and

$$s_{0m} = -j\cos\frac{m\pi}{2n} \qquad (18)$$

where

S_{0m} = location of m^{th} zero in the complex plane
$m = 1, 3, 5, \cdots 2n-1$.

With the left half-plane poles and zeros specified in (13) and (18), respectively, the reflection coefficient, $\rho(j\omega)$, can be formulated.

$$\rho(j\omega) = \frac{\prod_{m=1,3,5}^{2n-1}(s - s_{0m})}{\prod_{k=1,3,5}^{2n-1}(s - s_{pk})}. \qquad (19)$$

The driving point impedance, $Z_{11}(j\omega)$, can then be expressed in terms of the reflection coefficient, or

$$Z_{11}(j\omega) = \frac{1-\rho(j\omega)}{1+\rho(j\omega)}. \qquad (20)$$

A two-terminal-pair ladder network terminated at both ends can be synthesized from the driving-point impedance. The method of attack is dependent upon the desired circuit configuration. If the network is to have a shunt capacitor across the input terminals, a pole of the driving-point *admittance*, $Y_{11}(j\omega)$, must be removed at $s=\infty$. If a series inductance is desired at the input, a pole at $s=\infty$ must be removed from the driving-point *impedance*, $Z_{11}(j\omega)$. The expansion then continues until all of the components in the desired ladder network have been specified.

In a similar manner, a low-pass prototype can be synthesized such that its response function provides a "maximally-flat" (Butterworth) selectivity, or

$$|t(j\omega)|^2 = \frac{1}{1+\omega^{2n}}. \qquad (21)$$

Using (22), the left half-plane poles of the Butterworth function can be specified

$$s'_{pk} = -\sin\frac{k\pi}{2n} + j\cos\frac{k\pi}{2n}, \qquad (22)$$

where

s'_{pk} = location of the k^{th} pole in the complex plane,
$k = 1, 3, 5, \cdots 2n-1$,
n = order of the polynomial.

It should be noted that, in this case, the n zeros of $\rho(j\omega)$ all occur at $s=0$, or

$$s_{0m}' = 0, \qquad (23)$$

where

S'_{0m} = location of the $m_0'^{\text{th}}$ zeros in the complex plane,
$m = 1, 3, 5, \cdots 2n-1$.

With the left half-plane poles and zeros specified in (22) and (23), respectively, the reflection coefficient for the Butterworth approximation can also be formulated using (19).

Example

The design procedure which has been outlined in the previous paragraphs can best be illustrated with an example. In order to simplify the mathematics, it will be assumed that a three-stage network having resistive terminations at both ends and a 3 db ripple in the pass-band (*i.e.*, $\epsilon = 1$) is required.

With

$\alpha = \sqrt{2}$	from (16)
$\beta = 1$	from (17)
$\sinh\phi_2 = 0.298$	from (14)
$\cosh\phi_2 = 1.043$	from (15)

the poles of the network can be computed from (13),

$$s_{p1} = -0.149 + j0.903$$
$$s_{p3} = -0.298 + j0.$$
$$s_{p5} = -0.149 - j0.903.$$

From (18), the zeros are found to be

$$s_{01} = -j0.866$$
$$s_{03} = -j0$$
$$s_{05} = +j0.866.$$

Substituting the poles and zeros in (19),

$$\rho(j\omega) = \frac{s(s^2 + 0.750)}{s^3 + 0.596s^2 + 0.927s + 0.250}.$$

From (20),

$$Z_{11}(j\omega) = \frac{0.596s^2 + 0.177s + 0.250}{2s^3 + 0.596s^2 + 1.677s + 0.250}.$$

Since a shunt capacitance is desired at each terminal pair, the continued fraction will be formulated such that a pole is initially removed from the driving-point admittance, $1/Z_{11}(j\omega)$, at $s = \infty$.

$$\begin{array}{r|l} & 3.36s \\ .596s^2 + .177s + .250 & 2s^3 + .596s^2 + 1.677s + .250 \\ & 2s^3 + .596s^2 + \ .838s \\ \hline & \qquad\qquad .839s + .250 \end{array}$$

$$\begin{array}{r|l} & .71s \\ .839s + .250 & .596s^2 + \ .177s + .250 \\ & .596s^2 + \ .177s \\ \hline & \qquad\qquad .250 \end{array}$$

$$\begin{array}{r|l} & 3.36s \\ .250 & .839s + .250 \\ & .839s \\ \hline & \qquad .250 \end{array}$$

$$\begin{array}{r|l} & 1.0 \\ .250 & .250 \end{array}$$

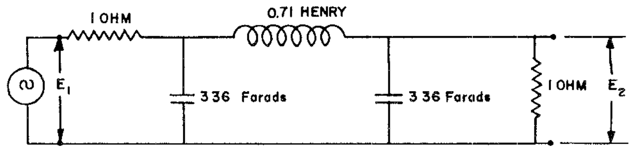

FIG. 13—Low-pass prototype ($\eta = 3$) having Tchebycheff response ($\epsilon = 1$).

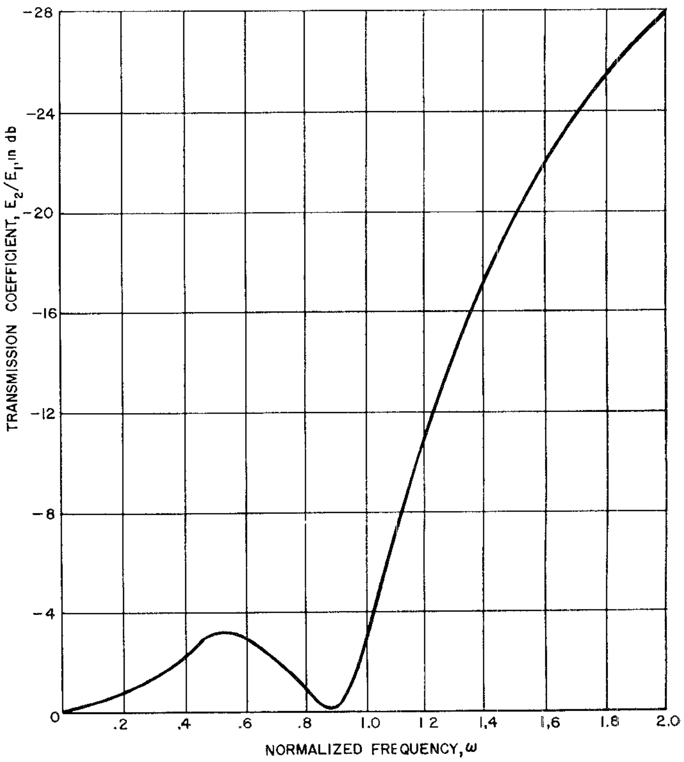

Fig. 14—Tchebycheff response ($\epsilon = 1$) characteristics for low-pass prototype ($\eta = 3$).

The resulting network and its selectivity are shown in Figs. 13 and 14, respectively.

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